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Connecting Spin and Statistics in Quantum Mechanics

Arthur Jabs

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Abstract The spin-statistics connection is derived in a simple manner under the postulates that the original and the exchange wave functions are simply added, and that the azimuthal phase angle, which defines the orientation of the spin part of each single-particle spin-component eigenfunction in the plane normal to the spin-quantization axis, is exchanged along with the other parameters. The spin factor $(-1)^{2s}$ belongs to the exchange wave function when this function is constructed so as to get the spinor ambiguity under control. This is achieved by effecting the exchange of the azimuthal angle by means of rotations and admitting only rotations in one sense. The procedure works in Galilean as well as in Lorentz-invariant quantum mechanics. Relativistic quantum field theory is not required.

Keywords Spin and statistics · Spinor · Spinor ambiguity · Bose and Fermi statistics · Pauli exclusion principle · Symmetrization

Physics is simple but subtle
Paul Ehrenfest

1 Introduction

The standard method of treating systems of identical particles in present quantum mechanics is to require that every wave function or state vector must be either symmetric

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A. Jabs (✉)
Alumnus, Technical University Berlin, Voßstr. 9, 10117 Berlin. Germany
e-mail: arthur.jabs@alumni.tu-berlin.de

or antisymmetric, that is, multiplied by either $+1$ or -1 when the labels or parameters referring to any two particles are interchanged. There are thus two classes of systems, with different collective behaviour of the particles: systems of bosons and systems of fermions. These two classes are connected with the spins of the particles: all particles which are known to be bosons are empirically found to have integral spin, in units of $\hbar$, while all known fermions have half-integral (i.e., half-odd-integral) spin.

Within quantum mechanics the connection with spin could not be derived and had to be taken as another postulate. The first derivation was provided by Fierz [1] and Pauli [2], who founded it on relativistic quantum field theory. This also remained the framework for the papers which in subsequent years refined and generalized Pauli's proof [3, 4]. Typically, in these papers it is postulated that no negative-energy states exist, that the metric in Hilbert space is positive definite, and that the fields either commute or anticommute for spacelike separations (locality, microcausality). Under these conditions it is shown that integral-spin fields cannot satisfy the (fermionic) anticommutation relations, and half-integral-spin fields cannot satisfy the (bosonic) commutation relations. This does not exclude the possibility that fields exist which satisfy other commutation relations and show statistics that differ from Bose and Fermi statistics.

In 1965 Feynman in his Lectures on Physics [5, p. 4-3] objected:

> An explanation has been worked out by Pauli from complicated arguments of quantum field theory and relativity. He has shown that the two [spin and statistics] must necessarily go together, but we have not been able to find a way of reproducing his arguments on an elementary level. It appears to be one of the few places in physics where there is a rule which can be stated very simply, but for which no one has found a simple and easy explanation. The explanation is deep down in relativistic quantum mechanics. This probably means that we do not have a complete understanding of the fundamental principle involved.

The aim of the present paper is to propose such a simple and easy explanation.

Actually, since 1965 more than a hundred publications appeared deriving the spin-statistics connection under different sets of conditions [6]. Reviews are contained in [7–10]. Many of these publications derive the connection in settings far removed from standard (local) relativistic quantum field theory; and they are also far from simple and easy.

Closest to the present approach are those papers that use only quantum mechanics, relativistic or nonrelativistic, and are written in the spirit of Feynman's demand for simplicity. These papers nevertheless contain one or several of the following restrictions: the wave functions must have special invariance [11, 12], continuity [13–17] or symmetry [18] properties, or must lie in special spin-component subspaces [19]. The systems considered must be nonrelativistic [13–22], have only two spatial dimensions [22], contain only two particles [18, 19], only particles with zero spin [13–17] or spin $\leq 1/2$ [20, 21], only point particles [23–25], must admit antiparticles [26], or the exchange must be considered as physical transportation of real objects [12, 23–25, 27].

The present proposal is not subject to any of these restrictions. It grew out of an attempt to understand the papers by York [28, 29] in the framework of the realist interpretation which I developed some time ago [30, 31].[1] The premises of the present proposal are seen when the organization of the paper is considered.

In Sect. 2 we start with Feynman's method of superposing transition amplitudes. This is shown to be equivalent to symmetrizing or antisymmetrizing only the final but not the initial wave function in the transition amplitudes. Moreover, in line with another proposal by Feynman, symmetrizing or antisymmetrizing the final wave function, which for two particles means addition or subtraction of the original and the exchange wave function, is replaced by the "addition postulate", which admits only addition.

The minus sign in the superposition of fermionic wave functions arises from the construction of the exchange function. In Sect. 3 we point out that it is important to consider the azimuthal spin angle χ, which defines the orientation of the spin part of a single-particle spin-component eigenfunction. The angle χ is also exchanged, but requires a special treatment because it expresses the well known spinor ambiguity when the spin component m is half-integral. This means that we cannot know which of the two possible values of the function with the exchanged χ has to be chosen.

In Sect. 4 it is shown that the ambiguity can be overcome by effecting the exchange of χ by way of rotations and by admitting only rotations in one direction, either clockwise or counterclockwise. In Sect. 5 the rotations leading from the original to the exchange function are explicitly carried out, and it is shown that the exchange function thereby acquires the desired spin factor $(-1)^{2s}$. It is thus the rotation group, a subgroup of both the Galilei and the Lorentz group, that determines the type of statistics. In standard quantum mechanics it is the permutation group that does this: its one-dimensional representations are associated with Bose and Fermi statistics, and its other representations with "parastatistics". The rotation group in our approach leads only to Bose and Fermi statistics, and there is no reason to suggest experiments in search of particles with parastatistics.

In Sect. 6 the proof is extended to N particles, and in Sect. 7 to particles with different spin components. Finally, in Sect. 8, using the properties of helicity functions, it is pointed out that the proof also holds in relativistic quantum mechanics.

2 Adding Up Transition Amplitudes

It is remarkable that wave-function symmetrization (or antisymmetrization) is never mentioned in the Feynman Lectures on Physics [5] and yet the same physical situations as in the other textbooks, which do use symmetrization, are acounted for.

[1] In this interpretation, which is opposed to the Copenhagen interpretation, the quantum objects are not point particles but extended objects, represented by the wave functions, like $\psi(\boldsymbol{r}^{(i)}) \equiv \psi^{(i)}(\boldsymbol{r})$. The index (i), rather than denoting the particle i in the wave function, denotes the wave function itself. The vector $\boldsymbol{r}^{(i)}$ does not mean the position of point particle i (where is it when its position is not being measured?) but the position variable of wave function i, and $|\psi(\boldsymbol{r}^{(i)})|^2 \mathrm{d}^3 r$ is the probability that the wave function i causes an effect about the point $\boldsymbol{r}$. In either interpretation, realist or Copenhagen, these labels are needed to avoid self-interactions or to show the one-particle operators which wave function to operate on. So, the construction of the proof presented here is independent of which of the interpretations is adopted, and the present paper sticks to the traditional formulation.

Feynman basically considers *transition amplitudes*: when two transitions cannot be distinguished in principle from each other, the amplitudes, rather than the probabilities, have to be superposed [5, pp. 1-10, 3-7,…,4-3]. The superposition includes the phase factor $\exp(\mathrm{i}\delta)$

$$f = f(\theta) + e^{\mathrm{i}\delta} f(\pi - \theta) \tag{1}$$

in Feynman's notation. In line with standard quantum mechanics the phase factor in the Lectures is eventually postulated to be $+1$ or -1.

What does this mean in terms of (Schrödinger) wave functions? The transition amplitude for the transition from $\Psi_\mathrm{a}(1,2)$ to $\Psi_\mathrm{b}(1,2)$ is given by the scalar product

$$f = (\Psi_\mathrm{b}(1,2), \Psi_\mathrm{a}(1,2)).$$

These scalar products are basic elements of quantum mechanics because any expression of physical significance, that is, an expression which yields the probability of a result of a measurement, is formulated in terms of absolute squares of transition amplitudes

$$|f|^2 = |(\Psi_\mathrm{b}, \Psi_\mathrm{a})|^2.$$

Now, in systems of identical particles the transition from $\Psi_\mathrm{a}(1,2)$ to $\Psi_\mathrm{b}(1,2)$ cannot be distinguished in principle from a transition from $\Psi_\mathrm{a}(1,2)$ to $\Psi_\mathrm{b}(2,1)$, where $\Psi_\mathrm{b}(2,1)$ denotes the exchange function, in which the labels or parameters referring to particles 1 and 2 have been exchanged. Thus, in terms of wave functions (1) takes on the form

$$\begin{aligned} f &= (\Psi_\mathrm{b}(1,2), \Psi_\mathrm{a}(1,2)) + e^{\mathrm{i}\delta}(\Psi_\mathrm{b}(2,1), \Psi_\mathrm{a}(1,2)) \\ &= (\Psi_\mathrm{b}(1,2) + e^{\mathrm{i}\delta}\Psi_\mathrm{b}(2,1), \Psi_\mathrm{a}(1,2)), \end{aligned} \tag{2}$$

and with $e^{\mathrm{i}\delta} = \pm 1$

$$f = (\Psi_\mathrm{b}(1,2) \pm \Psi_\mathrm{b}(2,1), \Psi_\mathrm{a}(1,2)). \tag{3}$$

Here only one of the two functions in the scalar product is a superposition of the original and the exchange function, and no normalization factors appear. This is the form that is exclusively used by Feynman.

This is to be compared with the standard method in quantum mechanics, where the amplitude in the above case is written as

$$f = \left(\frac{1}{\sqrt{2}}[\Psi_\mathrm{b}(1,2) \pm \Psi_\mathrm{b}(2,1)], \frac{1}{\sqrt{2}}[\Psi_\mathrm{a}(1,2) \pm \Psi_\mathrm{a}(2,1)]\right). \tag{4}$$

Here both functions in the scalar product are superpositions of an original and an exchange function, and both functions are normalized, by means of the factors $1/\sqrt{2}$.

Feynman's formula (3) and the standard formula (4) are equivalent provided the phase factor $\exp(\mathrm{i}\delta)$ is restricted to the values ± 1. The proof of this is based on properties of the symmetrizer S and the antisymmetrizer A,

$$S = (1/N!)\sum_\alpha P_\alpha, \qquad A = (1/N!)\sum_\alpha (-1)^{k_\alpha} P_\alpha,$$

where N is the number of identical particles ($N > 2$ here included), P_α is a permutation, and k_α is the number of transpositions (interchanges) that make up the permutation P_α. The operators S and A satisfy the relations

$$S^\dagger = S = S^2 \quad \text{and} \quad A^\dagger = A = A^2, \tag{5}$$

the proof of which can be found in [32, p. 1383–1385].

With these relations the standard formula (4) in the Fermi case, when written for N-particle functions

$$f = \left(\sqrt{N!}A\Psi_{\mathrm{b}}(1,2,\ldots,N), \sqrt{N!}A\Psi_{\mathrm{a}}(1,2,\ldots,N)\right) \tag{6}$$

becomes

$$f = N!(A^\dagger A\Psi_b, \Psi_a) = N!(A\Psi_b, \Psi_a) = \left(\sum_\alpha (-1)^{k_\alpha} P_\alpha \Psi_b, \Psi_a\right), \tag{7}$$

which is Feynman's formula (2) with $e^{\mathrm{i}\delta} = -1$, when written for N-particle functions. In the Bose case the equivalence is obtained in an analogous way.

When the factors in front of the P_αs in S and A, that is 1 and $(-1)^{k_\alpha}$, respectively, are replaced by more general factors inspection of the proof of relations (5) in [32] shows that the proof no longer goes through, and with this the equivalence between the standard formulas (4), (6) and the Feynman formulas (2), (3), (7) no longer holds. Such a more general factor appears however in the course of our treatment when it comes to one-particle wave functions with different spin components (η_α, formula (28)). We thus have to make a choice. It is Feynman's formulas that we choose because we found that in contrast to the standard method the Feynman method allows us to incorporate a derivation of the spin-statistics connection and yet finally to arrive at the approved expressions of physical significance.

Physically, Feynman's method recommends itself because it is really only necessary to consider physically significant expressions, and the scalar products are closer to these ($|f|^2$) than are the wave functions (cf. e.g. [33, 34]). In any case it appears reasonable to number the particles in the first function arbitrarily and, assuming that the numbering gets lost in the transition, to add up all possibilities of numbering in the second function. In (anti)symmetrizing only the second function the Feynman method is less restrictive than the standard one. This may be the deeper reason why the Feynman method allows for a derivation of the spin-statistics connection, while the standard methods does not.

In choosing the Feynman method we shall see that finally in all expressions of physical significance the more general factor η_α reduces to $(-1)^{2sk_\alpha}$. This allows us to write $e^{\mathrm{i}\delta} = (-1)^{2s}$ in Feynman's transition amplitudes (2) and (3), and $(-1)^{k_\alpha} = (-1)^{2sk_\alpha}$ in his amplitude (7). These factors are restricted to the values ± 1, like the factors in S and A. With this the equivalence between the standard and the Feynman method will be recovered, with the additional bonus that the Feynman method has enabled us to establish the connection with spin.

In his own attempt to derive the spin-statistics connection Feynman [27] moreover suggests that we may

> take the view that the Bose rule is obvious from some kind of understanding that the amplitude[s] in quantum mechanics that correspond to alternatives must be added.

That is, he is proposing to start a priori with $\exp(\mathrm{i}\delta) = +1$ everywhere. We follow also this proposal by Feynman and just add up the transition amplitudes. That is, we start from

$$f = (\Psi_\mathrm{b}(1,2) + \Psi_\mathrm{b}(2,1), \Psi_\mathrm{a}(1,2))$$

in the case of two particles. In the case of N particles this obviously generalizes to

$$f = \left(\sum_\alpha P_\alpha \Psi_\mathrm{b}(1,2,\ldots,N), \Psi_\mathrm{a}(1,2,\ldots,N)\right),$$

where P_α is a permutation of the parameters referring to the single particles. The sum extends over the $N!$ possible permutations, including the identity I. The sum $\sum_\alpha P_\alpha \Psi_\mathrm{b} = \Psi_{\mathrm{bS}'}$ is the extension of the sum of the original and the exchange function from two-particle to N-particle systems and corresponds to the symmetrized function in standard quantum mechanics.

3 The Special Parameter

In order to present the essential points in a simple way we begin by considering a non-relativistic (Galilean) system of two identical particles of spin s ($\boldsymbol{S}^2\psi = s(s+1)\psi$) described by a (Schrödinger) wave function which is a product of two normalized one-particle wave functions

$$\Psi = \psi^{(1)}(a,m)\psi^{(2)}(b,m),$$

and the one-particle wave functions are eigenfunctions of the operator of the spin component with respect to an arbitrary but common spin-quantization axis. Moreover, both functions belong to the same eigenvalue m. These restrictions will be removed in Sects. 6 to 8.

The single-particle wave functions are functions of the variables x, y, z, t. The functions are determined by the parameters a, b and m, where a and b stand for the sets of parameters that, together with m and χ (below), allow for a *complete* account of all aspects and degrees of freedom of the single-particle systems. Such a set of parameters includes e.g. mass, charge, total spin, centre, expansion coefficients etc. Mass, charge, and total spin are of course the same for identical particles and their exchange has no effect. An alternative notation would be $\psi(a, m, x^{(1)}, y^{(1)}, z^{(1)}, t)$, where the labels in parentheses, (1) and (2), distinguish the particles in the formalism. We have suppressed here the variables and have put the particle labels directly at the function symbols. In Sects. 2 to 6 the eigenvalue m is always the same and will also be omitted from the notation. Thus we write

$$\Psi = \psi^{(1)}(a)\psi^{(2)}(b). \tag{8}$$

Among the parameters of the wave function there is one that requires special treatment in the construction of the exchange function because it may lead to double-valued wave functions. The reason is that the spin parts of the wave functions, while belonging to one and the same m, may still differ from one another by a rotation about the spin-quantization axis. In other words, each spin part has a definite orientation in a plane normal to the common spin-quantization axis, defined by an azimuthal angle χ, counted from some arbitrary reference direction. A complete spin-quantization *frame* rather than only a spin-quantization *axis* is involved. The angle χ is kept out of the set a (and b) and is exhibited explicitly. Each function can have its own angle, but the particular values do not matter. The values of χ are restricted to the interval $[0, 2\pi]$.

The specific form of the parametric dependence of the spin-component eigenfunction on χ is given by the factor

$$\exp(\mathrm{i}m\chi), \tag{9}$$

so that

$$\psi^{(1)}(a, \chi_a) = \exp(\mathrm{i}m\chi_a)\psi^{(1)}(a). \tag{10}$$

The angle χ appears "only" in a phase factor, and when this factor is an overall (global) phase factor it is without any physical significance and can be omitted. However, it must be taken seriously if it is to become part of a superposition, thereby determining a *relative* phase and thus becoming physically significant [32, pp. 219, 220]. This is what happens in the present approach: the angle χ is exchanged, though in a specific way, along with the other parameters, and in Sect. 5, when the original and the exchange function are superposed, χ becomes instrumental in determining the relative phase between these functions.

The exponential factor (9) expresses the spinor ambiguity: in the case of half-integral m it is $+1$ for $\chi = 0$ and -1 for $\chi = 2\pi$. The sign change under a full rotation holds for every rotation axis. The factor (9) refers to the special case of rotations about the spin-quantization axis. This does not lead out of the subspace of functions with the same m. All this is standard quantum mechanics [32, pp. 694, 703, 985, 986].

Note that the rotation of the *orbital* angular momentum part of a wave function can be expressed by changing the value of an already present spatial variable ($\varphi \to \varphi + \varphi'$ [32, pp. 681, 682]), but that an additional parameter (χ) is needed to express the behaviour of the *spin* part of a wave function under a rotation.

The letter χ, rather than the customary φ, is used in order to emphasize that it is the spin part, not the orbital part, which is concerned, that the spin-quantization axis need not coincide with the z-axis, and that the angle χ is not a variable of the one-particle wave function, as $\boldsymbol{r}$ and φ are. Rather, the dependence on χ is a parametric dependence, like that on m and the other parameters in a and b. Therefore the application of a differential operator like $-\mathrm{i}\hbar\partial/\partial\chi$, analogous to the z-component of *orbital* angular momentum, does not make sense for the spin part of the wave function.

Thus, with the explicit appearance of χ (8) becomes

$$\Psi = \psi^{(1)}(a, \chi_a)\psi^{(2)}(b, \chi_b). \tag{11}$$

In constructing the exchange wave function in traditional quantum mechanics it is irrelevant whether we exchange the particle labels (1), (2) or the function parameters a, b. In our construction it is no longer irrelevant, and it is the exchange of the parameters that must be chosen. Thus we replace a by b and vice versa in the original wave function (11). But because the parameter χ expresses the spinor ambiguity the exchange of χ_a with χ_b and vice versa cannot be done in such a simple way.

4 Controlling the Spinor Ambiguity

The special feature with the factor (9) is that χ is an angle, so that we may go from some particular value χ_a to some other value χ_b in two ways, either clockwise or counterclockwise. In the case of half-integral m one way leads to a different wave function at χ_b than the other, the two functions having different signs. In other words, the value of the function at χ_b then depends not only on the value of χ_b but also on the path leading from χ_a to χ_b. This leads to double-valued functions and represents another aspect of the spinor ambiguity.

One may imagine the function $\exp(\mathrm{i}m\chi)$ with half-integral m to lie on the two-sheeted Riemannian surface of the function $\sqrt{z}$ [35, 36], where one sheet carries only one set of function values. The clockwise path from χ_a to χ_b always ends up in a different sheet than the counterclockwise path. Or one may imagine a Möbius band, where on the first round trip over the band one set of function values is met, and the corresponding other set on the second round trip. In fact, devices like twisted ribbon belts [27, p. 58], contortions of an arm holding a cup [27, p. 30] and others [37–42][2] are similar to the Riemannian surface and the Möbius band in that they construct an indicator of whether we are in the first or in the second turn, and in that they return to the original situation after the second turn. For integral m (including $s = m = 0$) the Riemannian surface has only one sheet and no ambiguity arises.

Now, when adding the original and exchange wave functions the functions must be uniquely defined. This is not the same as the general requirement that wave functions be single-valued. Single-valuedness can only be required for measurable quantities such as transition probabilities or expectation values, but not for the wave functions themselves [43]. In many textbooks it is nevertheless invoked for the wave functions themselves, in particular for justifying the restriction to integral values of m for orbital angular momentum. The real justification of integral m here rests on group representations and properties of observables [44].

Our case is different because we are concerned with the procedure of constructing one wave function by superposition of others, formally similar to interference. The demand for removing the spinor ambiguity is in line with the demand for removing the ambiguity known as exchange degeneracy, that is, to the fixation of the coefficients in the superpositions in Sect. 2.

Now, according to what has been said above the spinor ambiguity is removed (i.e., kept under control) if we make a choice between the two possible paths from χ_a to

[2] And literature cited in these references.

χ_b, that is, if we exchange the χs by way of rotations and decide to make all rotations in one sense only, either clockwise or counterclockwise.

In the language of group theory the clockwise and the counterclockwise way from χ_a to χ_b correspond to paths of different homotopy classes (e.g. [45]). So our choice means that we are admitting only paths of the same homotopy class.

5 Constructing the Exchange Function

We are now ready to take the decisive step. We want to construct the exchange function from the original function (11), not by simply replacing χ_a by χ_b in the wave function $\psi^{(1)}$ and χ_b by χ_a in the wave function $\psi^{(2)}$ (as is done with the other parameters, a and b), but by continuously rotating the spin part of the functions from χ_a to χ_b and from χ_b to χ_a respectively, with due consideration being given to the paths connecting χ_a and χ_b.

Thus, we start from formula (11) where the as and bs have already been exchanged, but the χs have not:

$$\psi^{(1)}(b, \chi_a)\psi^{(2)}(a, \chi_b). \tag{12}$$

We then rotate the function $\psi^{(1)}(b, \chi_a)$ from χ_a to χ_b. We take the counterclockwise sense of the rotations, and we assume $\chi_a < \chi_b$ and $m \geq 0$. In order to get from χ_a to χ_b we then have to run through $\chi_b - \chi_a$. This yields the rotation factor $\exp(\mathrm{i}m(\chi_b - \chi_a))$ and we obtain

$$\psi^{(1)}(b, \chi_b) = e^{\mathrm{i}m(\chi_b - \chi_a)}\psi^{(1)}(b, \chi_a). \tag{13}$$

Likewise, rotating the function $\psi^{(2)}(a, \chi_b)$ counterclockwise from χ_b to χ_a means that we have to run through $2\pi - (\chi_b - \chi_a)$. This yields the rotation factor $\exp(\mathrm{i}m(2\pi + \chi_a - \chi_b))$ and we obtain

$$\psi^{(2)}(a, \chi_a) = e^{\mathrm{i}m(2\pi + \chi_a - \chi_b)}\psi^{(2)}(a, \chi_b). \tag{14}$$

Inserting (13) and (14) into (12) then yields the exchange function

$$F \times \psi^{(1)}(b, \chi_b)\psi^{(2)}(a, \chi_a) \tag{15}$$

with

$$F = e^{-\mathrm{i}m(\chi_b - \chi_a)}e^{-\mathrm{i}m(2\pi + \chi_a - \chi_b)} = e^{-\mathrm{i}m2\pi} = (-1)^{2m} = (-1)^{2s}, \tag{16}$$

where for the last equality we have used the fact that s and m are either both integral or both half-integral. Had we chosen the clockwise sense we would have obtained $F = \exp(+\mathrm{i}m2\pi)$, which is also equal to $(-1)^{2s}$. The same result obtains when $m < 0$ or when $\chi_a > \chi_b$. The case $\chi_a = \chi_b$ is of statistical weight zero and can be neglected.

Finally, adding the original function (11) and the exchange function (15) we arrive at

$$\Psi_\mathrm{S} = \psi^{(1)}(a, \chi_a)\psi^{(2)}(b, \chi_b) + (-1)^{2s}\psi^{(1)}(b, \chi_b)\psi^{(2)}(a, \chi_a). \tag{17}$$

Ψ_S need not be normalized in Feynman's method, as emphasized after formula (3). The angle χ and the rotations become effective only in the procedure of exchanging the parameters of the wave functions. In this procedure the angles χ_a and χ_b in the original and the exchange wave function are related in such a way that, although χ_a and χ_b may be randomly distributed in the original function, the resultant factor, $(-1)^{2s}$, is independent of χ_a and χ_b. Thus, once exchange and addition are accomplished, we may, according to formula (10), write the functions in (17) in the form $\psi^{(1)}(a, \chi_a) = \exp(\mathrm{i}m\chi_a)\psi^{(1)}(a)$ etc. Either term in (17) thereby receives the same factor

$$\exp(\mathrm{i}m[\chi_a + \chi_b]), \tag{18}$$

which can be put as an overall phase factor in front of the parentheses. There it can be omitted, that is, absorbed in the general arbitrary phase factor connected with every wave function.

Thus we are returning to the standard form of the wave functions, which do not exhibit the dependence on χ:

$$\Psi_S = \psi^{(1)}(a)\psi^{(2)}(b) + (-1)^{2s}\psi^{(1)}(b)\psi^{(2)}(a). \tag{19}$$

There is some formal analogy with interference between two parts of a split wave. One part is left unmodified [wave function (11)], the other is subject to a phase shift [exchange, wave function (15)], and then the two are recombined [wave function (17) or (19)].

When (19) is inserted into Feynamn's formula (2) with $\Psi_b(1,2) + e^{\mathrm{i}\delta}\Psi_b(2,1) =: \Psi_S$, we see that $e^{\mathrm{i}\delta} = (-1)^{2s}$, and with this we have reached our goal for the considered class of functions: the factor $(-1)^{2s}$ is no longer postulated but is derived in a simple way from basic principles. This factor yields $+1$ (bosons) for integral s and -1 (fermions) for half-integral s, and this is the desired connection between spin and statistics.

6 General Case. Equal Spin Components

We begin now to remove the restrictions imposed on the wave function in the previous sections. In the present section we first remove the restriction to two particles and then to functions of product form. The N-particle functions of product form are

$$\Psi_b = \psi^{(1)}(u_{r_1}, \chi_{t_1})\psi^{(2)}(u_{r_2}, \chi_{t_2})\cdots\psi^{(N)}(u_{r_N}, \chi_{t_N})$$

with $u_{r_1}, u_{r_2}, \ldots$ instead of a, b. The $\psi^{(i)}(u_{r_i}, \chi_{t_i})$ all belong still to the same m, which is therefore dropped from the notation. The symmetrized function is

$$\Psi_{bS'} = \sum_\alpha P_\alpha \psi^{(1)}(u_{r_1}, \chi_{t_1})\cdots\psi^{(N)}(u_{r_N}, \chi_{t_N}). \tag{20}$$

The index S′ (with prime) is to indicate that P_α in (20) permutes the parameter sets u_{r_i} among the one-particle functions but does not permute the angles χ_{t_i}. The permutation of the angles will be effected separately, by way of rotations. As any permutation can be written as a product of a number of transpositions, the term $P_\alpha \psi^{(1)}(u_{r_1}, \chi_{t_1}) \cdots \psi^{(N)}(u_{r_N}, \chi_{t_N})$ differs from the term with $P_\alpha = I$ by a number k_α of transpositions. When the χ rotations are applied, as described in the preceding sections for the case of two particles, every single transposition yields the factor $F = (-1)^{2s}$ in front of the term with interchanged parameters, independent of the angles χ. Hence k_α transpositions yield the factor $(-1)^{2sk_\alpha}$. The function (20) then changes into the superposition function (symmetric or antisymmetric)

$$\Psi_{\mathrm{bS}} = \sum_\alpha (-1)^{2sk_\alpha} P_\alpha \psi^{(1)}(u_{r_1}, \chi_{t_1}) \cdots \psi^{(N)}(u_{r_N}, \chi_{t_N}). \tag{21}$$

The index S (without prime) instead of S′ (with prime, as in (20)) is to indicate that the exchange of the angles χ_{t_i} by means of rotation is included. In (21) P_α therefore permutes the pairs $\{u_{r_i}, \chi_{t_i}\}$. The single functions $(-1)^{2sk_\alpha} P_\alpha \psi^{(1)}(u_{r_1}, \chi_{t_1}) \cdots \psi^{(N)}(u_{r_N}, \chi_{t_N})$ for $P_\alpha \neq I$ are the extensions of the exchange function from two to N particles.

As in the case of two particles, the one-particle functions in (21) may now be written as $\psi^{(i)}(u_{r_i}, \chi_{t_i}) = \exp(\mathrm{i}m\chi_{t_i})\psi^{(i)}(u_{r_i})$. This yields the same factor

$$\exp\left(\mathrm{i}m \sum_{k=1}^{N} \chi_{t_k}\right)$$

in front of every permutation operator P_α in (21). This factor can thus be drawn out of the permutation sum and becomes again an overall phase factor, which can be omitted. We can thus drop the χ_{t_i}s from the notation in (21) and simply write

$$\Psi_{\mathrm{bS}} = \sum_\alpha (-1)^{2sk_\alpha} P_\alpha \psi^{(1)}(u_{r_1}) \cdots \psi^{(N)}(u_{r_N}). \tag{22}$$

Now, if s is an integer, then $(-1)^{2sk_\alpha} = +1$ for any k_α, and Ψ_{bS} is symmetric (bosonic). If s is a half-integer, then $(-1)^{2sk_\alpha} = -1$ for odd k_α, and $+1$ for even k_α, and Ψ_{bS} is antisymmetric (fermionic). And this holds for each one of the $2s+1$ values of m.

Now we remove the restriction to wave functions of product form, but still with equal ms. The general N-particle function then is

$$\Phi_{\mathrm{b}} = \sum_{r_1,\ldots,r_N,t_1,\ldots,t_N} a_{r_1\cdots r_N t_1\cdots t_N} \psi^{(1)}(u_{r_1}, \chi_{t_1}) \cdots \psi^{(N)}(u_{r_N}, \chi_{t_N}), \tag{23}$$

where the sum over the rs and ts goes over a possibly infinite number of values. Permuting the parameter sets u_{r_i} among the one-particle functions and permuting the angles χ_{t_i} by way of rotations now results in

$$\Phi_{\mathrm{bS}} = \sum_{r_1,\ldots,r_N,t_1,\ldots,t_N} b_{r_1\cdots r_N t_1\cdots t_N} \sum_\alpha (-1)^{2sk_\alpha} P_\alpha \psi^{(1)}(u_{r_1}, \chi_{t_1}) \cdots \psi^{(N)}(u_{r_N}, \chi_{t_N}). \tag{24}$$

The sums $\sum_\alpha$ may be considered as basis functions in the subspace of wave functions with equal ms. They are symmetric or antisymmetric. The antisymmetric ones are special Slater determinants, on which the Pauli exclusion principle is based.

From the consideration in the preceding section (cf. (18)) it follows that the permutation sum $\sum_\alpha$ in (24) can be written as

$$\exp(\mathrm{i}m[\chi_{t_1} + \cdots + \chi_{t_N}]) \sum_\alpha (-1)^{2sk_\alpha} P_\alpha \psi^{(1)}(u_{r_1}) \cdots \psi^{(N)}(u_{r_N}). \tag{25}$$

So

$$\Phi_{\mathrm{bS}} = \sum_{r_1,\ldots,r_N,t_1,\ldots,t_N} c_{r_1\cdots r_N t_1\cdots t_N} \sum_\alpha (-1)^{2sk_\alpha} P_\alpha \psi^{(1)}(u_{r_1}) \cdots \psi^{(N)}(u_{r_N}), \tag{26}$$

where the coefficients $c_{r_1\cdots t_N}$ are thought to have absorbed the exponentials in (25). The coefficients thus include the angles χ_{t_i}. This is no problem because any superposition with definite coefficients means that the arbitrary phase factors, which accompany every wave function, have been fixed. This includes that the angles χ_{t_i}, which are arbitrary anyway, are fixed when they are part of those arbitrary phases. With this they produce no longer any effect. Note also that the sum over the rs and ts in (26), in contrast to the sum over α, has nothing to do with the identity of the particles in the final state.

The connection between spin and statistics is thus proved for general nonrelativistic N-particle functions composed of one-particle functions with the same spin component.

7 Different Spin Components

When admitting one-particle wave functions which belong to different spin components m, the factor F in (16) of Sect. 5 has to be replaced by a different factor, F_χ. This factor is obtained when the transposition procedure of Sect. 5 between (12) and (16) is repeated with (12) being replaced by $\psi^{(1)}(b, m_b, \chi_a)\ \psi^{(2)}(a, m_a, \chi_b)$. The result is

$$F_\chi = (-1)^{2s} \exp(-\mathrm{i}(m_a - m_b)(\chi_a - \chi_b)). \tag{27}$$

As any permutation P_α can be written as a product of a number k_α of transpositions, the factor $(-1)^{2sk_\alpha}$ in (21) to (26) is then replaced by η_α, which is a product of F_χs:

$$\eta_\alpha = (-1)^{2sk_\alpha} \prod^{(\alpha)} \exp\left(-\mathrm{i}(m_a - m_b)(\chi_a - \chi_b)(1 - \delta_{k_\alpha 0})\right). \tag{28}$$

The values that the parameters m_a, χ_a, m_b, χ_b assume in the various factors of the product are those of the particular one-particle functions on which the transpositions in P_α operate. The factor $(1 - \delta_{k_\alpha 0})$ with the Kronecker delta is to make the product 1 when $k_\alpha = 0$, that is, when $P_\alpha = I$. As we shall see, the exact form of η_α does not matter. What matters is first that in the case of equal ms the factor F_χ reduces to F, and η_α to $(-1)^{2sk_\alpha}$, so that we are back to Sects. 2 to 6, and second that in any case

$|\eta_\alpha| = 1$. What we have to show is that even in the case of different ms the factor η_α is effectively $(-1)^{2sk_\alpha}$ and the angles χ will disappear from the final expressions of physical significance.

The general N-particle wave functions now are (cf. (23))

$$\Phi_\text{a}(1,\ldots,N) = \sum_{\substack{r_1,\ldots,r_N\\ s_1,\ldots,s_N\\ t_1,\ldots,t_N}} a_{r_1\cdots r_N s_1\cdots s_N t_1\cdots t_N}\psi^{(1)}(u_{r_1}, m_{s_1}, \chi_{t_1})\cdots\psi^{(N)}(u_{r_N}, m_{s_N}, \chi_{t_N}), \tag{29}$$

$$\Phi_\text{b}(1,\ldots,N) = \sum_{\substack{\rho_1,\ldots,\rho_N\\ \sigma_1,\ldots,\sigma_N\\ \tau_1,\ldots,\tau_N}} b_{\rho_1\cdots\rho_N\sigma_1\cdots\sigma_N\tau_1\cdots\tau_N}\psi^{(1)}(u_{\rho_1}, m_{\sigma_1}, \chi_{\tau_1})\cdots\psi^{(N)}(u_{\rho_N}, m_{\sigma_N}, \chi_{\tau_N}), \tag{30}$$

where the sum over the s_is and σ_is goes over the $2s+1$ possible values of the spin component. We then have

$$\begin{aligned}\Phi_\text{bS} &= \sum_\alpha \eta_\alpha P_\alpha \Phi_\text{b}(1,\ldots,N)\\ &= \sum_{\rho_1,\ldots,\tau_N} b_{\rho_1\cdots\tau_N}\sum_\alpha \eta_\alpha P_\alpha \psi^{(1)}(u_{\rho_1}, m_{\sigma_1}, \chi_{\tau_1})\cdots\psi^{(N)}(u_{\rho_N}, m_{\sigma_N}, \chi_{\tau_N}).\end{aligned} \tag{31}$$

With a view to an expression of physical significance we consider the transition amplitude

$$f = (\Phi_\text{bS}, \Phi_\text{a}) = \sum b^*_{\rho_1\cdots\tau_N}\sum a_{r_1\cdots t_N} T_{(rst\rho\sigma\tau)}, \tag{32}$$

where

$$\begin{aligned}T_{(rst\rho\sigma\tau)} = \sum_\alpha \eta^*_\alpha P_\alpha(&\psi^{(1)}(u_{\rho_1}, m_{\sigma_1}, \chi_{\tau_1}), \psi^{(1)}(u_{r_1}, m_{s_1}, \chi_{t_1}))\\ &\cdots(\psi^{(N)}(u_{\rho_N}, m_{\sigma_N}, \chi_{\tau_N}), \psi^{(N)}(u_{r_N}, m_{s_N}, \chi_{t_N})),\end{aligned} \tag{33}$$

and the P_α permute only the sets $\{\rho_i, \sigma_i, \tau_i\}$, not the sets $\{r_i, s_i, t_i\}$, among the one-particle functions. The point is that the one-particle functions with different spin components are mutually orthogonal, irrespective of the us and χs. Thus, the scalar products between these functions are all proportional to Kronecker deltas, and the terms (33) become

$$T_{(rst\rho\sigma\tau)} = \sum_\alpha \eta^*_\alpha P_\alpha \kappa_{\sigma_1 s_1}\cdots\kappa_{\sigma_N s_N}\delta_{\sigma_1 s_1}\cdots\delta_{\sigma_N s_N}, \tag{34}$$

where

$$\kappa_{\sigma_i s_i} = \left(\psi^{(i)}(u_{\rho_i}, m_{\sigma_i}, \chi_{\tau_i}), \psi^{(i)}(u_{r_i}, m_{s_i}, \chi_{t_i})\right).$$

The only non-zero terms in the sum $\sum_\alpha$ are those where in each Kronecker delta the pair of indices consists of equal numbers, $\sigma_i = s_i$ for each i, although different Kroneckers may have different pairs.

There are three types of these terms:

(i) Terms where all σ_is, and hence all s_is, are equal. These cases are those already solved in the preceding sections, yielding the desired result $\eta_\alpha = (-1)^{2sk_\alpha}$.

(ii) Terms where all σ_is, and hence all s_is, are different ($N \leq 2s+1$). In these terms the sum $\sum_\alpha$ over the permutations reduces to one single member, where P_α is the identity I, with $k_\alpha = 0$ and $\eta_\alpha = 1$. We may therefore formally write $\eta_\alpha = (-1)^{2sk_\alpha}$, which again conforms with the desired result. The term T then reduces to

$$T_{(rst\rho\sigma\tau)} = (\psi^{(1)}(u_{\rho_1}, m_{s_1}, \chi_{\tau_1}) \cdots \psi^{(N)}(u_{\rho_N}, m_{s_N}, \chi_{\tau_N}),$$
$$\psi^{(1)}(u_{r_1}, m_{s_1}, \chi_{t_1}) \cdots \psi^{(N)}(u_{r_N}, m_{s_N}, \chi_{t_N})) = (\Psi_\mathrm{b}, \Psi_\mathrm{a}), \quad (35)$$

where Ψ_a and Ψ_b are product functions like those in (29) and (30) but without the sums, and in the special case $\sigma_i = s_i$.

In the special subcase that the product functions Ψ_a and Ψ_b already represent the total wave functions the transition amplitude (32) is equal to (35), that is, to one term only. The same then holds for the *probability* of the transition, $|(\Psi_\mathrm{b}, \Psi_\mathrm{a})|^2$, and there are no interference terms involving different transitions. This is as in a system of distinguishable particles. We have here a generalization of a well known result for two particles [5, p. 3-12], [32, pp. 1407, 1408], [46].

(iii) Terms which consist of two sets with equal σ_is and hence equal s_is within each set: $\sigma_1 = \sigma_2 = \cdots = \sigma_l \neq \sigma_{l+1} = \sigma_{l+2} = \cdots = \sigma_N$. Then those members of the sum (34) where the permutations work only within one set can be treated like those in point (i) and yield the desired result, whereas those members where the permutations exchange parameters of the first set with those of the second are zero. – Cases of more than two such sets are only technically more complicated but add nothing essential.

Thus in all expressions of physical significance the general factor η_α of (28) can be replaced by the factor $(-1)^{2sk_\alpha}$, in which there is no longer any dependence on χ and which can only assume the values ± 1. Thus finally we see that indeed the standard and the Feynman method are equivalent, as announced in Sect. 2.

Another point deserves to be mentioned: the term $T_{(rst\rho\sigma\tau)}$ of formula (34) is zero if only one single-particle function, referring to a particular particle, has a different spin component in the first than in the second total wave function. The terms (34) thus seem to be restricted to spin-independent interactions. Spin-dependent (or whatever dynamical) interactions can, however, thought to be included by considering the possible wave functions that result from such an interaction as intermediate functions Ψ_i between Ψ_a and Ψ_bS, where we have to form the sum over the product of the amplitudes $\sum_i (\Psi_\mathrm{bS}, \Psi_i)(\Psi_i, \Psi_\mathrm{a})$ in the case that the (situations associated with the) Ψ_i are not observed, or over the product of the respective probabilities in the case that the Ψ_i *are* observed.

8 The Relativistic Domain

The derivation of the spin-statistics connection presented so far evidently does not require relativity theory. Can it be extended into the relativistic domain? In Lorentz-invariant theory spin and orbital angular momentum are no longer separately conserved quantities, and the two are in general mixed up in a complicated way. There are however functions which are eigenfunctions of the spin-component operator only, with no admixture of orbital angular momentum: the helicity functions [47]. A helicity function describes a free particle with definite non-zero linear momentum and is an eigenfunction of the operator of the spin component with respect to an axis that is parallel or antiparallel to the direction of the particle's momentum. Thus we may replace the previously discussed eigenfunctions of the operator of the spin component along a fixed direction by the helicity functions. Helicities are invariant under ordinary rotations (involving spin and orbital part), and the rotation operators commute with the permutation operators, so we may express the momentum eigenfunctions which have their momenta in arbitrary directions by suitably rotated eigenfunctions with momenta in one common direction (cf. [47, pp. 407, 408]). For these functions we can define a common reference direction for the angles χ, and then construct and add up the functions with the permuted parameters in the previously described way. This works not only for momentum eigenstates, i.e. plane waves, but also for linear superpositions of plane waves, i.e. wave packets.

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